The adjoint filter operator in large-eddy simulation of turbulent flow

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Adjoint and self-adjoint filter operators are introduced, such that large-eddy simulation (LES) with a spatially variable filter width satisfies important physical properties: Conservation of momentum and dissipation of kinetic energy. The combination of an arbitrary nonuniform explicit filter with the Smagorinsky model leads to a new model of the turbulent stress tensor, which includes backscatter, while the total subgrid dissipation is still positive (analytically). Nonuniform filter theory is further developed, in order to provide a more solid foundation of practical LES. The paper distinguishes between three sets of equations: The Navier–Stokes equations (which are physical conservation laws), the filtered equations and the modeled large-eddy equations. It is shown that general filtering of the Navier–Stokes equations destroys their local and global conservation properties. However, it is proven that the adjoint of a normalized filter is conservative. As a result, the filtering equations are globally conservative, for special nonuniform (e.g., self-adjoint) filters. Implications for six subgrid-models that require explicit filter operations are considered, such as dynamic, similarity, filtering multiscale, and relaxation models. Incorporation of the adjoint filter analytically ensures several models to conserve momentum and dissipate kinetic energy. Examples of adjoint and self-adjoint filters are also provided, including a ‘‘three-points’’ self-adjoint filter and an adjoint filter that is applicable on unstructured grids. In addition, it is shown that positive nonuniform (self-adjoint) filters satisfy mathematical smoothing properties. The focus is on kernel filters, but projection filters are also discussed, and nonuniform self-adjoint Laplace filters are defined. The (orthogonal) projection operator is proven to be a nonuniform kernel filter. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1710479]

I. INTRODUCTION

Consider the incompressible Navier–Stokes equations in a bounded domain Ω with appropriate boundary conditions and initial conditions:

$$\frac{\partial u_j}{\partial x_j}=0, \qquad \frac{\partial u_i}{\partial t}+\frac{\partial u_i u_j}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\upsilon\frac{\partial^2 u_i}{\partial x_j^2}, \tag{1}$$

where u is the velocity, p the pressure, and υ the constant kinematic viscosity. The equations above are physical conservation laws, globally and locally. Global conservation means that the total mass and momentum in the domain is constant, assuming periodic or zero Dirichlet boundary conditions. The divergence form of the equations also implies local conservation within an arbitrary volume, provided the fluxes through the volume face are taken into account.

The basic equations in large-eddy simulation (LES) are derived by the application of a filter to the Navier–Stokes equations,

$$\overline{f}=Gf. \tag{2}$$

For the time being, we only assume that the filter is linear and commutes with the time derivative. Thus the filtered equations read

$$\overline{\frac{\partial u_j}{\partial x_j}}=0, \qquad \frac{\partial \overline{u_i}}{\partial t}+\overline{\frac{\partial u_i u_j}{\partial x_j}}=-\overline{\frac{\partial p}{\partial x_i}}+\upsilon\overline{\frac{\partial^2 u_i}{\partial x_j^2}}. \tag{3}$$

We will distinguish between the filtered equations and the modeled large-eddy equations, which are formulated in Sec. III.

In most cases, the filter is an integral operator with a specified filter kernel. The earliest filter is the top-hat filter (e.g., Deardorff[1]). In fact Reynolds[2] introduced this averaging operator over a three-dimensional, spatial, rectangular region. In large-eddy simulation, a filter width is associated with the filter, often proportional to the local grid-spacing, which defines a separation of the turbulence into resolved and subgrid scales. Leonard[3] generalized the top-hat filter operation to a general convolution integral, admitting other filter kernels, but with a uniform filter width. This filter commutes with spatial derivatives[3] and the resulting filtered equations are still local conservation laws, because they can be written in divergence form.

Next to the filtering approach, Schumann[4] proposed a procedure, which respects the local conservation laws, also

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for nonuniform averaging volumes. A more recent approach separates resolved and subgrid scales with a projection, using a set of general basis functions (e.g., Ref. 5). In this context, Pope[5] imposed an important constraint on his modeled equations: The (global) conservation of momentum. Although the focus of the current paper is on ''kernel'' filters, the formulation will be so general that the filter (2) can also represent a projection operator.

In practical applications, it is often desirable to have a nonuniform filter, which is a filter that depends on the spatial location. In wall-bounded flows, for example, a relatively small filter width is required in the boundary layer, in order to resolve the near wall structures responsible for the turbulent production. If the filter width varies, there is no general commutation between filter and spatial derivatives in the filtered equations.[6–13] For a uniform filter, commutation is lost near a solid boundary because the support of the filter does not remain inside the domain.[7,9]

We note that, due to the commutation problem, the filtered equations are in general not local conservation laws. Thus, general nonuniform filtering destroys an important physical property of the Navier–Stokes equations. However, common large-eddy simulations usually employ discretizations of conservation laws and do not reckon with the nonconservative character of the commutator error. Indeed, one might argue that, rather than modeling a quantity that is not conservative, the filtered equations should be recast as a form which is at least globally conservative.

In this paper we will, therefore, show that nonuniform filters can be constructed which ensure global conservation. Then the filtered equations resemble important physical properties of the Navier–Stokes equations. In actual large-eddy simulations the filter does not always explicitly occur in the modeled equations and then the filter is mainly needed to interpret the results. However, many subgrid models do involve explicit filter operations. In particular if the explicit filter is taken on the vector-level, global conservation becomes relevant in practice.

Furthermore, an essential physical feature of turbulence is its dissipative character. The kinetic energy is cascaded from large to small scales and a subgrid-model should therefore drain energy from the resolved scales. To increase the robustness of practical LES is another motivation to adopt dissipative subgrid models. For these reasons, eddy-viscosity models were introduced, the Smagorinsky model[14] being the most famous one. In this paper we will investigate how several existing models that employ an explicit filter can be reformulated, such that the dissipation of kinetic energy can analytically be proven.

For the purposes above, a counterpart of an arbitrary filter will be introduced: the adjoint filter, which will be proven to be globally conservative (Sec. II). Section III will consider six subgrid-models involving explicit test-filter operations. For several models, conservation of momentum and dissipation of kinetic energy become analytical properties, due to the inclusion of the adjoint filter. Examples of adjoint and self-adjoint filter operators will be constructed and some of them will be applicable to unstructured meshes. Related topics, such as nonkernel filters, smoothing behavior and boundary conditions are discussed in Sec. V. Conclusions will be drawn in Sec. VI.

II. THE ADJOINT FILTER

Consider a general filter, defined as a linear spatial operator by Eq. (2). The following natural requirement is usually imposed on the filter:

$$Gc=c \quad \text{where} \quad \forall x\in\Omega: \quad c(x)=1, \tag{4}$$

which states that the filter does not alter a constant field. Such a filter is *normalized*. In addition, we call the filter *conservative*, if it does not alter the integral of an arbitrary function f on Ω:

$$\forall f: \quad \int_\Omega Gf(x)\,dx=\int_\Omega f(x)\,dx. \tag{5}$$

With a conservative filter, the filtered equations (3) remain globally conservative, because then

$$\int_\Omega \overline{\frac{\partial f}{\partial x_j}}\,dx=\int_\Omega \frac{\partial f}{\partial x_j}\,dx=0, \tag{6}$$

for appropriate boundary conditions. If the volume integral is the appropriate representation of the ensemble average then the conservative filter is statistically consistent, that is $\langle\overline{f}\rangle=\langle f\rangle$.

To introduce the adjoint filter, we first write the standard innerproduct for functions $\Omega\to\mathbb{R}$ in the Hilbert space $L_2(\Omega)$:

$$(f,g)=\int_\Omega f(x)g(x)\,dx. \tag{7}$$

By definition, the *adjoint* operator G^a, corresponding to G, satisfies[15]

$$\forall f,g: \quad (G^a f,g)=(f,Gg). \tag{8}$$

One can prove that a normalized filter leads to a conservative adjoint filter:

$$\int_\Omega G^a f(x)\,dx=(G^a f,c)=(f,Gc)=(f,c)=\int_\Omega f(x)\,dx. \tag{9}$$

Reversely, a conservative filter has a normalized adjoint, since Eq. (5) implies

$$\forall f: \quad (G^a c,f)=(c,Gf)=\int_\Omega Gf(x)\,dx$$
$$=\int_\Omega f(x)\,dx=(c,f), \tag{10}$$

which implies $G^a c=c$. Thus a normalized filter and a conservative adjoint filter are equivalent [note that $(G^a)^a$ equals G].

The filter is *self-adjoint* if $G^a=G$. Evidently, a normalized self-adjoint filter is conservative, which leads to globally conservative filtered equations. In fact, a normalized and conservative filter also provides conservative equations. Such a filter is not necessarily self-adjoint, as will be shown by an example in Sec. IV.

The remaining part of this section concerns kernel filters. The most general expression for a nonuniform spatial kernel filter is[16] (compare the product filter in Ref. 6)

$$Gf(x)=\int_\Omega K_G(x,\xi)f(\xi)d\xi, \tag{11}$$

where $K_G:\Omega\times\Omega\rightarrow\mathbb{R}$ is the filter function and x,ξ are locations in the three-dimensional flow domain Ω. A space-dependent filter width $\Delta(x)$ can be associated with this kernel filter [see Appendix A for some possible definitions of $\Delta(x)$]. We assume that

$$\int_\Omega\int_\Omega|K_G(x,\xi)|^2 d\xi dx<\infty, \tag{12}$$

which implies that the filter is a bounded operator.

For kernel filters, the normalization property (4) is equivalent to a normalized filter function:

$$\forall x\in\Omega:\quad \int_\Omega K_G(x,\xi)d\xi=1, \tag{13}$$

a well-known property for common filters in large-eddy simulation. The conservation property (5) is equivalent to

$$\int_\Omega Gf(x)dx=\int_\Omega\int_\Omega K_G(x,\xi)f(\xi)d\xi dx = \int_\Omega K_G(x,\xi)dx \quad \int_\Omega f(\xi)d\xi. \tag{14}$$

Consequently, a conservative filter is equivalent to

$$\forall \xi\in\Omega:\quad \int_\Omega K_G(x,\xi)dx=1. \tag{15}$$

Note that the conservation property of the filter function (15) differs from the normalization (13) property. Many nonuniform filters do not satisfy Eq. (15). Thus, the corresponding filtered equations are in general not globally conservative.

Standard integral operator theory implies that the adjoint of kernel filter is also a kernel filter. Its filter function equals

$$K_{G^a}(x,\xi)=K_G(\xi,x), \tag{16}$$

which is proven by

$$(G^a f,g)=\int_\Omega\left[\int_\Omega K_G(\xi,x)f(\xi)d\xi\right]g(x)dx = \int_\Omega f(\xi)\left[\int_\Omega K_G(\xi,x)g(x)dx\right]d\xi=(f,Gg). \tag{17}$$

Obviously, the kernel filter is self-adjoint if the filter function is symmetric in its arguments:

$$\forall x,\xi\in\Omega:\quad K_G(x,\xi)=K_G(\xi,x). \tag{18}$$

A normalized self-adjoint kernel filter is conservative indeed, because Eqs. (13) and (18) imply Eq. (15). This verifies that the filtered equations are globally conservative in case of a normalized self-adjoint kernel filter.

In the special case of a convolution filter C, the kernel is defined as $K_C(x-\xi)$. This standard filter is uniform and the normalization property implies conservation. Equation (18) shows that the convolution filter is self-adjoint, if the kernel is even in its single argument $x-\xi$. Linear operator theory states that all eigenvalues of a self-adjoint operator are real. The eigenvalues of a convolution filter are the values of the Fourier transfer of the kernel, while the corresponding eigenfunctions are Fourier waves. Generalizing this, the "transfer function" of a nonuniform filter can be defined by the eigenvalues of G. If Ω is bounded and K_G is finite and continuous then G is a compact operator. According to the Hilbert–Schmidt theorem[15] the eigenfunctions of G form an orthonormal basis of $L_2(\Omega)$.

Examples of nonuniform filters are the standard top-hat and Gaussian filter, with the uniform Δ replaced by the nonuniform $\Delta(x)$. Ghosal and Moin[7] introduced a class of nonuniform filters by defining the convolution integral in computational space. Also filters can be constructed[8,11] which reduce the commutation error to an arbitrarily high order term in Δ. Nevertheless, a particular realization of such a filter has a fixed order and then the approximate commutation does not imply exact conservation of mass and momentum.

All these examples of nonuniform filters satisfy the normalization property. In each case, the adjoint filter is well defined by Eq. (16). The examples above are not self-adjoint, because Eq. (18) is not fulfilled, and are not conservative either [Eq. (15) does not hold, at least not exactly]. It is interesting that Schumann's volume averaging operator[4] is equivalent to a piecewise constant top-hat filter, which is nonuniform, conservative and self-adjoint. Other exactly conservative, self-adjoint kernel filters will be constructed in Sec. IV.

III. SUBGRID MODELS

The theoretical properties of the adjoint filter extend the possibilities of several subgrid models to conserve momentum and dissipate kinetic energy. We will analyze six models that involve explicit filter operations in actual large-eddy simulations: two dynamic models, two filtering multiscale models, a relaxation model and the similarity model. However, we first introduce the modeled large-eddy equations, discuss their position with respect to the filtered equations and formulate the standard Smagorinsky model.

In most actual large-eddy simulations, these *modeled large-eddy* equations are solved on Ω:

$$\frac{\partial w_j}{\partial x_j}=0, \qquad \frac{\partial w_i}{\partial t}+\frac{\partial w_i w_j}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 w_i}{\partial x_k^2}-R_i, \tag{19}$$

where the boundary conditions are usually the same as for the Navier–Stokes equations. The difference with the Navier–Stokes equations is the extra term R_i, which represents the subgrid-model. As indicated for physical reasons it is required that, with appropriate boundary conditions, R_i

conserves momentum and dissipates kinetic energy. By construction, numerical methods (e.g., finite volume methods) often conserve mass and momentum. In some cases, like in the MILES approach,[17] the numerical scheme takes the dissipation of the subgrid turbulence into account.

Apparently, practical LES does not solve the filtered equations (3), but an approximation of the Navier–Stokes equations (19). The term R_i is constructed such that the solution w contains a smaller range of scales than the solution u of the Navier–Stokes equations (1). In order to interpret w, the basic filter operation, which leads to the filtered equations (3), can be helpful. In case the magnitude of R_i is substantial, the solution w will never represent all features of u, especially not the small-scale phenomena. However, it might be reasonable to require that w approximately represents the large scales contained in u. The large scales are extracted by a formal, basic filter, which defines $\overline{u}$.

The formal filter can not only be used for the interpretation of w, but also occurs, through the filtered equations, in the definition of the subgrid terms that actually should be modeled by R_i. These definitions are important, since their mathematical structure can inspire the modeling; consider for example the similarity and gradient models. Common definitions distinguish between two subgrid terms: the commutator and the divergence of the turbulent stress. If the basic filter is conservative, the commutator is (globally) conservative and, consequently, the sum of the two subgrid terms is conservative. In addition, the dissipation of kinetic energy caused by R_i can be interpreted as effects of both the commutator error and the standard turbulent stress tensor. These observations support the definition of a new subgrid term, into which the commutators and the divergence of the turbulent stress are lumped together.

The earliest subgrid model employed in large-eddy simulation, is the Smagorinsky eddy-viscosity model:[14]

$$
\begin{aligned}
R_i &= \frac{\partial}{\partial x_j} m_{ij}(w), \\
m_{ij}(w) &= -2C_S^2\Delta^2 |S(w)| S_{ij}(w), \\
S_{ij}(w) &= \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right), \\
|S(w)|^2 &= 2S_{ij}(w)S_{ij}(w).
\end{aligned}
\tag{20}
$$

Here C_S is the model parameter and Δ is the basic filter width, often equal to the local grid-spacing.

The Smagorinsky model does not need any explicit definition of the filter before it can be used in LES, although it would be quite interesting to know which (possibly probabilistic) operator relates the predictions by the model to the physical velocity. The Smagorinsky model is in divergence form, thus locally conservative, and has the following global properties with appropriate boundary conditions:

$$
\epsilon_R = \int_\Omega w_i R_i dx \geqslant 0 \quad \text{and} \quad \int_\Omega R_i dx = 0, \tag{21}
$$

which expresses that the model dissipates kinetic energy and conserves momentum. Appropriate boundary conditions are for example a no-slip condition for w to let the boundary integral vanish in the partial integration of ϵ_R. The integrated R_i is zero if m_{ij} is not active at the boundary, which is the case if, for example, $\Delta = 0$ at the boundary.

However, several models exist which incorporate explicit filtering operations in their evaluations. The explicit filter in the well-known dynamic model (Germano *et al.*[18]) is the test-filter. This model only modifies the model coefficient in the Smagorinsky model and, consequently, the dynamic model is momentum conserving. It is guaranteed to dissipate kinetic energy, but only if the dynamic coefficient is positive. This is usually achieved with by a somewhat *ad hoc* clipping procedure, which simply puts the eddy-viscosity to zero at locations where the dynamic procedure produces a negative value.

Recently, a dynamic model using the vector level identity has been proposed by Morinishi and Vasilyev,[19] which takes effects of the commutator into account through a different determination of the dynamic coefficient. Like the standard dynamic model, this model is in divergence form, which implies that the modeled large-eddy equations (19) are locally conservative. However, the other set of equations, the filtered equations (3), is in general not conservative.

Better agreement between the dynamically modeled equations and the filtered equations is achieved, if the basic filter is normalized and conservative, because then both sets of equations are (globally) conservative. In order to meet the similarity assumption at different filter levels in the dynamic procedure, test-filter and basic filter should ideally have a similar form. Consequently, the application of a normalized and conservative (e.g., self-adjoint) test-filter is desirable, for the sake of consistency.

The same argument holds for the explicit filter in the generalized similarity model below (35). Similarity is not assumed for the multiscale and relaxation models in this section. They are consistent and have the desired analytic properties for any normalized nonuniform filter. However, even then the application of a self-adjoint operator has the practical advantage that only one filter needs to be implemented, because the filter and its adjoint counterpart are the same.

In the four models that follow, the explicit filter does not alter the model coefficient, but the structure of the model itself. The original formulations are modified and the adjoint operator of the explicit filter is incorporated. Two filtering multiscale models are considered, one in divergence form. In the case of wall-bounded flows, several multiscale models[20–23] can compete with the dynamic model, whereas the standard Smagorinsky model without wall-damping is not accurate in such flows.

The principle of multiscale models, which were introduced into LES by Hughes *et al.*,[24] is that the subgrid model only depends on the smallest resolved scales (see also Guermond[25] and Layton[26]). The underlying reasoning is the physical energy cascade; the energy transfer to subgrid scales is mainly caused by the smallest resolved scales, and not so much by the largest scales. The Smagorinsky model has in particular been successful for homogeneous isotropic turbulence. For this reason its behavior in inhomogeneous flows is expected to improve, if it is applied after the largest (inhomogeneous) scales have been omitted from the velocity field. A limitation of multiscale models is that it neglects the so-called "spectral eddy-viscosity plateau," if the explicit filter is in the inertial range. To take this aspect of turbulence into account, it may be necessary to model the large-scale equation as well.[22]

The largest and smallest resolved scales are split by an explicit filter, G:

$$\begin{aligned} f&=Gf+Ff,\\ Ff&=(I-G)f,\\ F^af&=(I-G^a)f, \end{aligned} \tag{22}$$

where $If=f$, the large scales are defined by Gf and the small ones by Ff. Note that the normalization of G implies that G^a is conservative and, therefore, the integral of F^af equals zero. It is also remarked that in this section, the symbol G represents the explicit filter in models, often not equal to the basic filter. The explicit filter width in multiscale models is usually proportional to Δ, for example, 2Δ, but it could also be a fraction of a certain large-scale L.

Multiscale models were first proposed in a weak, variational formulation,[24] involving scale separation by a projection operator. Dissipation of kinetic energy was proven.[24] One of the models in this variational multiscale method is the "small-small" model, which is the Smagorinsky model entirely expressed in the small resolved scales. This variational multiscale method, including its equations for the large- and small-scales in the resolved velocity, can be extended to scale separation by a general filter within a strong formulation.[22] One of the multiscale models tested in the filtering analog reads

$$R_i=\frac{\partial}{\partial x_j}[Fm_{ij}(Fw)]. \tag{23}$$

Omitting one of the F's in this model also provides acceptable simulation results.[22] Model (23) can only be proven to dissipate kinetic energy if G satisfies Eq. (18) and commutes with the spatial derivative.[22] To remove these limitations, the adjoint operator can be used to formulate two modified versions, in order to prove dissipation of kinetic energy for an arbitrary nonuniform filter.

The first modification of the filtering multiscale model (23) loses the divergence form and, therefore, local conservation momentum. It is obtained if the first F is replaced by F^a and put before the divergence:[26]

$$R_i=F^a\left(\frac{\partial}{\partial x_j}[m_{ij}(Fw)]\right). \tag{24}$$

This model is equivalent to the variational multiscale model,[24] if a variational form and a projection operator are adopted. A variational form expresses each term as an innerproduct with a test-function. In that case it is not necessary to know F^a, since the innerproduct translates the action of F^a to an action of F. For this reason, the adjoint small-scale extraction (F^a) remained unknown in Ref. 26; at the end a variational form was proposed. The adjoint filter (G^a) did not occur at all. In the present paper we are interested in a direct evaluation of model (24). This is now possible, since F^a has become available, by the adjoint filter G^a, through Eq. (22).

Remark that model (24) does in general not conserve global momentum. However, this property is assured if G is normalized, since then the integral of F^af is zero. The dissipation of model (24) equals[24,26]

$$\begin{aligned} \epsilon_R&=-\int_\Omega w_iF^a\left(\frac{\partial}{\partial x_j}[m_{ij}(Fw)]\right)dx\\ &=-\int_{\partial\Omega}(Fw_i)m_{ij}(Fw)n_j dA\\ &\quad+\int_\Omega\frac{\partial(Fw_i)}{\partial x_j}m_{ij}(Fw)dx. \end{aligned} \tag{25}$$

The boundary integral over the surface $\partial\Omega$ of the domain Ω vanishes, if Fw or m_{ij} equals zero on the boundary. For the Smagorinsky base model, Eq. (25) reduces to

$$\epsilon_R=\int_\Omega C_S^2\Delta^2|S(Fw)|^3dx\geqslant 0. \tag{26}$$

Obviously, model (24) is dissipative for any dissipative base model m_{ij}.

The second modified version of the filtering multiscale model is in divergence form, unlike Eq. (24). For this purpose, the small-scale extraction operator F is not applied to the velocity but to the rate of strain:

$$\begin{aligned} R_i&=-\frac{\partial}{\partial x_j}F^a(2C_S^2\Delta^2|s|s_{ij}),\\ s_{ij}&=F(S_{ij}(w)),\\ |s|^2&=2s_{ij}s_{ij}. \end{aligned} \tag{27}$$

The definition of the adjoint operator (8) and partial integration proves positive dissipation for arbitrary G:

$$\begin{aligned} \epsilon_R&=-\int_\Omega w_i\frac{\partial}{\partial x_j}[F^a(2C_S^2\Delta^2|s|s_{ij})]dx\\ &=\int_\Omega S_{ij}(w)F^a(2C_S^2\Delta^2|s|s_{ij})dx\\ &\quad-\int_{\partial\Omega}w_in_jF^a(2C_S^2\Delta^2|s|s_{ij})dA\\ &=\int_\Omega 2s_{ij}C_S^2\Delta^2|s|s_{ij}dx=\int_\Omega C_S^2\Delta^2|s|^3dx\geqslant 0, \end{aligned} \tag{28}$$

provided the boundary term is zero. This is the case if the velocities are zero on the boundary. Model (27) involves ten filter operations, four more than Eq. (24) (and not twelve, since S_{ij} is symmetric and trace-free). There is of course no analytical limitation to replace the eddy-viscosity $2C_S^2\Delta^2|s|$ in model (27) by any other eddy-viscosity.

As model (27) is in divergence form, it corresponds to a new model of the standard turbulent stress tensor:

$$\tau_{ij}=-F^a(2C_S^2\Delta^2|s|s_{ij}). \tag{29}$$

This model is a symmetric tensor and includes backscatter, a well-known physically realistic feature of turbulence. Backscatter is defined by locally negative regions of[27–29]

$$\Pi = -\tau_{ij} S_{ij}(w). \tag{30}$$

The integral of Π equals the total subgrid dissipation ϵ_R, which is positive. The total backscatter relative to ϵ_R was about 13% in an LES that used model (29) and simulated the channel flow described in Ref. 22.

Model (27) is possibly the first subgrid model in literature that combines backscatter with an analytically positive subgrid dissipation, without prescribing a specific numerical method or *ad hoc* clipping. We remark that the standard definition of backscatter[27] is valid only if the model is in divergence form. Consider, for example, Eq. (24) and change τ_{ij} to the expression between square brackets. Then the integral of Π is no longer equal to the subgrid dissipation ϵ_R. The variational multiscale method[24] has a similar problem with the definition of backscatter.

Another advantage of the divergence form (27) is that the corresponding modeled equations resemble the locally conservative character of the Navier–Stokes equations. However, the nonuniformly filtered equations are not in divergence form, but only globally conservative for appropriate filters. From this point of view, a divergence form of the subgrid model is not mandatory; global conservation is sufficient.

The next subgrid model that involves explicit filtered operations is the relaxation model.[5,30] In combination with a deconvolution model, accurate results for channel flow were reported by Stolz *et al.*[30] The following relaxation term is found in Ref. 30:

$$R_i = \chi(I - GG_N) w_i. \tag{31}$$

The operator G_N is an $(N+1)$-terms standard geometric series expressed in F, consequently

$$R_i = \chi\left(I - G\frac{I - F^{N+1}}{I - F}\right) w_i = \chi F^{N+1} w_i. \tag{32}$$

Next, this model is formulated in terms of the adjoint filter:

$$R_i = (F^a)^n (\chi F^n w_i), \tag{33}$$

where $2n$ equals $N+1$. For a normalized G, the integral of $F^a f$ is zero, which implies that the reformulated relaxation term globally conserves momentum, even for spatially varying $\chi \geq 0$. Also it dissipates kinetic energy for arbitrary nonuniform G:

$$\epsilon_R = \int_\Omega w_i (F^a)^n (\chi F^n w_i) dx = \int_\Omega (F^n w_i) \chi F^n w_i dx \geq 0. \tag{34}$$

Here the definition of the adjoint operator (8) has been applied n times.

The last subgrid model is the generalized similarity model, introduced by formula (9) in Ref. 30, which reduces to

$$R_i = G\left(\frac{\partial w_i w_j}{\partial x_j}\right) - \frac{\partial}{\partial x_j}[(Gw_i)(Gw_j)], \tag{35}$$

for $N=0$. This expression becomes globally conservative if the normalized G is replaced by its adjoint counterpart G^a, in the first term only, or in the entire equation. No operator needs to be replaced if G is self-adjoint. If commutation is assumed and the last term in Eq. (35) is replaced by $w_i w_j$, the classic Leonard term[3] is recovered. In fact, Eq. (35) includes a model for the commutator error, because it equals a similarity model of the commutator and the divergence of the standard similarity model, which was proposed by Bardina *et al.*[28,29,31]

Linear combinations of the models listed above can be considered. In case of a similarity model plus one of the dissipative models above, a so-called mixed model is obtained.[22,28–31] Another linear combination is the sum of two dissipative models. As an example we mention Eq. (27) plus (a small fraction of) the standard Smagorinsky model. As the two components individually satisfy global conservation of momentum and dissipation of kinetic energy, the sum also has these analytical properties. A somewhat similar linear combination was already proposed by Schumann,[4] where one component accounted for inhomogeneous effects and the other for locally isotropic turbulence. However, dissipation of kinetic energy was not an analytical property yet.

In the future, compared to linear dissipations, nonlinear dissipative subgrid models, like Eqs. (24) and (27), will possibly be more attractive from a theoretical point of view, for the following reason. Existence and uniqueness of solutions has been proven by Ladyzhenskaya[32] in case of the Navier–Stokes equations plus the Smagorinsky model (see, e.g., Ref. 33 for more explanation). Until now, such a proof has not been delivered for the pure Navier–Stokes equations. An essential element in this important proof is that the Smagorinsky model is nonlinear in the velocity.

IV. CONSTRUCTION OF ADJOINT AND SELF-ADJOINT FILTERS

In this section three filters will be constructed. First, a continuous self-adjoint filter is derived, starting from an arbitrary normalized filter. Then a compact adjoint filter that is applicable to arbitrary meshes will be constructed. Finally a compact self-adjoint filter, valid for orthogonal meshes, will be proposed. The summation convention for repeated indices will not be used in this section.

For an arbitrary normalized filter G, not necessarily a kernel filter, we define the linear operator

$$Hf = Gf + \frac{1}{V}\int_\Omega [f(y) - Gf(y)] dy, \qquad V = \int_\Omega dx. \tag{36}$$

Then H is both normalized, $Hc = c$ [Eq. (4)], and conservative,

$$\int_\Omega Hf(x)dx=\int_\Omega Gf(x)dx+\left(\frac{1}{V}\int_\Omega dx\right)\times\left(\int_\Omega f(y)dy-\int_\Omega Gf(y)dy\right)=\int_\Omega f(y)dy. \quad (37)$$

This does not imply that H is self-adjoint, but a self-adjoint filter is now easily found:

$$J=\tfrac{1}{2}(H+H^a). \quad (38)$$

The filter J is self-adjoint, i.e., $J^a=J$ because $(H^a)^a=H$ for any operator H. Since H is normalized and conservative, H^a is conservative and normalized. Consequently, J has these two properties as well.

For nonuniform kernel filters, the filter functions of H and J can be derived, after the definition

$$b(\xi)=\frac{1}{V}\int_\Omega K_G(y,\xi)dy. \quad (39)$$

The normalization of G implies that the integral of b over Ω equals one. The filter functions corresponding to H and J are

$$K_H(x,\xi)=K_G(x,\xi)+\frac{1}{V}-b(\xi), \quad (40)$$

$$K_J(x,\xi)=\frac{1}{2}K_G(x,\xi)+\frac{1}{2}K_G(\xi,x)+\frac{1}{V}-\frac{1}{2}b(x)-\frac{1}{2}b(\xi). \quad (41)$$

The support of the self-adjoint filter J is obviously not compact. As actual implementations of filters are always discrete, we directly proceed with the discrete formulation in our construction of compact filters.

To construct such local filters, we first assume a general unstructured grid, where i denotes the index of the nodes and B is the set of indices of all nodes. The control volumes around grid nodes are denoted by Ω_i and V_i is the volume of Ω_i. The set of all Ω_i forms a partitioning of Ω.

A general normalized discrete filter is defined by

$$(Gf)_i=\sum_{j\in B}\alpha_{ij}V_jf_j. \quad (42)$$

The normalization constraint implies

$$\sum_{j\in B}\alpha_{ij}V_j=1. \quad (43)$$

In order to create local filters, "neighbor" sets B_i are defined. For each i, B_i contains N_i indices j, such that node j in physical space is identical or close to node i:

$$B_i=\{j|j=i \ \text{or}\ j\ \text{and}\ i\ \text{are close in physical space}\}. \quad (44)$$

The cases

$$\alpha_{ij}=\begin{cases}1\Big/\left(\sum_{m\in B_i}V_m\right) & \text{if}\ j\in B_i\\ 0 & \text{otherwise}\end{cases} \quad (45)$$

and

$$\alpha_{ij}=\begin{cases}1/(N_iV_j) & \text{if}\ j\in B_i\\ 0 & \text{otherwise}\end{cases}, \quad (46)$$

are two examples of local filters. The corresponding filter width is defined in Appendix A.

The adjoint filter corresponds to the transpose of the matrix α_{ij}:

$$(G^af)_i=\sum_{j\in B}\alpha_{ji}V_jf_j. \quad (47)$$

For this purpose the innerproduct

$$(f,g)=\sum_{i\in B}V_if_ig_i, \quad (48)$$

is adopted and the proof reads

$$(G^af,g)=\sum_{i\in B}V_i\left(\sum_{j\in B}\alpha_{ji}V_jf_j\right)g_i=\sum_{j\in B}\sum_{i\in B}V_jf_j(\alpha_{ji}V_ig_i)=(f,Gg). \quad (49)$$

In addition, using the normalization constraint (43), G^a is proven to be conservative:

$$\sum_{i\in B}V_i(G^af)_i=\sum_{i\in B}V_i\left(\sum_{j\in B}\alpha_{ji}V_jf_j\right)=\sum_{j\in B}V_jf_j\left(\sum_{i\in B}\alpha_{ji}V_i\right)=\sum_{j\in B}V_jf_j. \quad (50)$$

The last term is the discrete equivalent of the integral of f.

For the construction of self-adjoint filters, we turn to an orthogonal grid. In that case a three-dimensional filter is usually defined by the subsequent application of three "one-dimensional" filters. For this reason the following construction is restricted to one dimension. Thus, the location of the nodes are at x_i and the "volumes" equal

$$V_i=\tfrac{1}{2}(x_{i+1}-x_{i-1}). \quad (51)$$

Introducing a constant γ, with $0\leqslant\gamma\leqslant 1$, we consider the following nonuniform three-points filter:

$$\begin{aligned}&\alpha_{i,j}=0 \ \text{if}\ |i-j|\geqslant 2,\\ &\alpha_{i,i-1}=\frac{x_i-x_{i-1}}{2V_{i-1}V_i}(1-\gamma),\\ &\alpha_{i,i}=\gamma/V_i,\\ &\alpha_{i,i+1}=\frac{x_{i+1}-x_i}{2V_iV_{i+1}}(1-\gamma).\end{aligned} \quad (52)$$

This filter is self-adjoint, because the matrix α_{ij} is symmetric. The filter is normalized because Eq. (43) can be derived from Eq. (52) and, consequently, this self-adjoint three-points filter is conservative. The constant γ determines the local filter width (formulate Appendix A for the discrete case).

In case of nonperiodic boundary conditions, the definitions of $\alpha_{1,1}$ and $\alpha_{N,N}$ may have to be changed. Assume, for example, that the left boundary is a wall. If the normal coordinate, x, equals zero at the wall then

$$V_1=\tfrac{1}{2}(x_2+x_1), \tag{53}$$

where the first grid point, x_1, is either on the wall or in the interior of Ω. The first diagonal coefficient is determined by the normalization constraint:

$$\alpha_{1,1}=\frac{1-\alpha_{1,2}V_2}{V_1}. \tag{54}$$

V_N and $\alpha_{N,N}$ are defined in a similar way.

Suppose we have a k-points filter in one dimension with $k=2m+1$, centered around x_i (m points at each side). Outside the band of k diagonals, every value in the matrix α_{ij} equals zero. Then the adjoint filter employs k-points as well. A k-points normalized and conservative filter can be obtained by finding a solution to a linear system of $3N$ equations with kN unknowns. Here N denotes the total number of grid-points in one dimension. The $3N$ equations results from the requirements of normalization and conservation in each point, and the prescription of the filter width in each point. A self-adjoint filter would be a solution of $2N$ equations with $(m+1)N$ unknowns (use the symmetry property). Obviously, both solutions are not unique if $k>3$. An analogous procedure can be developed to find continuous compact self-adjoint filters, but the mathematics will become much more technical than for the discrete case.

Note that the volumes V_j in Eq. (42) have not been lumped into the filter coefficients α_{ij}, in order to make the analogy between continuous and discrete filters more clear. In this way, discrete self-adjoint filters correspond to symmetric matrices.

V. DISCUSSION

In this section several topics related to the previous sections will be discussed. First we will consider filters that are primarily not spatial integral operators: Projection filters, Laplace filters and temporal filters. Afterwards, two characterizations of the smoothing behavior of a filter will be proven and analyzed for several filters. Finally, we will briefly discuss which boundary conditions should be imposed in large-eddy simulation.

An alternative to the common approach of LES is projection-based LES.[5,24] The function space on Ω is spanned by an infinite set of basis functions $v_1,v_2,\ldots,$ whereas the projection operator P projects a signal on a finite set of basis functions $v_1,\ldots,v_N$. A projection operation in the context of LES can be regarded as a filter, here called "projection filter":

$$Gf=Pf=\Sigma_{k=1}^{N}\alpha_k v_k, \tag{55}$$

where α_k are the basis-function coefficients. Requirements of normalization and conservation give additional constraints on the basis-function coefficients.[5]

It is remarkable that an arbitrary orthogonal projection operator P can be written as a self-adjoint kernel filter. An orthonormal basis implies

$$\alpha_k=\int_\Omega f(\xi)v_k(\xi)d\xi. \tag{56}$$

Substitution of these coefficients into (55) yields

$$\begin{aligned}Pf(x)&=\sum_{k=1}^{N}\left(\int_\Omega f(\xi)v_k(\xi)d\xi v_k(x)\right)\\&=\int_\Omega\left[\sum_{k=1}^{N}v_k(\xi)v_k(x)\right]f(\xi)d\xi.\end{aligned} \tag{57}$$

The filter kernel $K_P(x,\xi)$ equals the expression between square brackets. As K_P is symmetric in its arguments, P is self-adjoint. Normalization and conservation are, therefore, equivalent. In fact, integral kernels can be derived for arbitrary bounded linear operators, including nonorthogonal projections. For this purpose the Riesz representation theorem[15] should be applied for each $x\in\Omega$.

The reformulation of projections as nonuniform kernel filters, directly implies that projection operators do in general not commute with derivatives. However, the assumption of commutation is not needed if the closure problem is redefined. See Refs. 5, and 24 (projection methods) and Eq. (22) in Ref. 22 (general filtering).

Next, we define nonuniform "Laplace" filters, which are always self-adjoint. The second-order term in the Taylor expansion of Gf for a top-hat or Gaussian convolution filter is the Laplace operator:[3,34]

$$Gf=f+\frac{\Delta_k^2}{24}\frac{\partial^2 f}{\partial x_k^2}+O(\Delta^4). \tag{58}$$

Here Δ_i is the filter width in the i-direction. A nonuniform Laplace filter can be defined by

$$Gf=f+\frac{1}{24}\frac{\partial}{\partial x_k}\left(\Delta_k^2\frac{\partial f}{\partial x_k}\right). \tag{59}$$

This filter is normalized, conservative and also self-adjoint. The latter is shown by partial integration, where boundary terms vanish, if either the filter width or the normal derivative of f is zero on the boundary. Replacing the plus by a minus sign and taking a uniform Δ, Eq. (59) becomes the inverse operator of the differential (or Helmholtz) filter, proposed by Germano.[35]

We considered spatial filters in this paper, but Sec. II can be generalized to filters with a temporal dimension. For this purpose, the innerproduct needs to be extended to four dimensions, including the time direction. Then adjoint and self-adjoint filters can be defined, but a complication is that the adjoint operator of a causal filter, which at a given time t_1 only depends on $t\leqslant t_1$, depends on the future ($t\geqslant t_1$).

The second subject of this section concerns the essential purpose of a filter; the filter should smooth out a fluctuating signal to some extent. To characterize the smoothing behavior of a nonuniform filter, theoretical smoothing properties are analytically derived and discussed for several filter types. The first one states that a filter does not increase the global maximum, neither decrease the global minimum of a variable:

$$\forall f: \quad \min_\Omega(f) \leqslant Gf \leqslant \max_\Omega(f). \tag{60}$$

The second smoothing property,

$$\forall f: \quad \int_\Omega (Gf)^2 dx \leqslant \int_\Omega f^2 dx, \tag{61}$$

means that the L_2-norm of a signal is not increased by filtering. This property implies that the norm of a normalized, self-adjoint G (the largest eigenvalue of G), is precisely one. Another implication of (61) is that the kinetic energy in the filtered field is smaller than in the unfiltered field.

For normalized nonuniform filters inequality (60) holds if the filter function is positive, as shown in Appendix B. If such a kernel filter is conservative, the second smoothing property (61) holds as well (Appendix B). That appendix also proves that inequality (61) is valid for orthogonal projection operators. The first smoothing property is not always satisfied for projection filters. As an example we mention the Fourier cut-off projection, which corresponds to a nonpositive kernel filter. Consequently, it may increase the global extrema of a variable (compare the well-known Gibbs phenomenon). The nonuniform Laplace filter does generally not satisfy the smoothing properties above. However, the properties are satisfied by the discrete version, provided all corresponding coefficients α_{ij} are positive. Using the standard seven points discrete Laplacian, this implies that Δ should be smaller than about 3.4 times the grid-spacing.

Finally, we discuss the boundary conditions, for example at a solid wall. The boundary conditions for f and Gf are the same if Δ approaches zero near the wall (Ghosal and Moin[7]). Boundary conditions for normal derivatives may also be required. Normal derivatives of f and Gf are the same in general, only if Δ equals zero in an arbitrarily small interval $[0,\delta]$ with $\delta > 0$. More specifically, G should be equal to the identity operator I in this interval, otherwise the original boundary conditions and those of the filtered equations are possibly different.

In practice the modeled equations (19) are always solved by imposing for the modeled velocity w_i just the physical boundary conditions of u_i. This does not necessarily exclude the application of an explicit filter that is not exactly zero at the wall. As indicated in Sec. III, similarity between the explicit filter and the (theoretical) basic filter is assumed for the dynamic and similarity models only. In case of the dynamic model there is no need to require the theoretically desired similarity in the direct vicinity of the wall, since there the dynamic coefficient usually equals zero.

In actual large-eddy simulations, the flow is often well resolved close to the wall which implies a locally small grid spacing and, consequently, small values of the normal filter width and its normal derivative. In that case, boundary conditions of u and Gu are approximately the same and the issue is not very important.

VI. CONCLUSIONS

We have considered theoretical properties of nonuniform filters and models explicitly incorporating such filters. In order to ensure that large-eddy simulations retain important physical properties of the Navier–Stokes equations, a framework has been developed in which the conservation of momentum and the dissipation of kinetic energy are essential. In this framework, the modeled equations in large-eddy simulation inherit these important physical properties of the original Navier–Stokes equations.

The adjoint filter has been introduced for a general nonuniform filter operator, which is a filter that allows a spatially variable filter width. If the filter is a kernel filter, that is an integral operator with kernel $K_G(x,\xi)$, then the kernel of the adjoint filter equals $K_G(\xi,x)$. A normalized filter G was proven to be equivalent to a conservative adjoint counterpart G^a. A filter is conservative is conservative if it does not change the integral of an arbitrary signal.

Unlike the Navier–Stokes equations, the nonuniformly filtered Navier–Stokes equations are in general not conservation laws. It was shown that, for general filters, global conservation is not satisfied either. However, normalized and conservative (e.g., self-adjoint) filters do result in globally conservative filtered equations. Then the filtered equations resemble an essential physical feature of the Navier–Stokes equations.

In practice, the modeled equations in LES are the Navier–Stokes equations supplemented with a subgrid model. It is important to distinguish between these equations and the (formal) filtered equations. The filtered equations mainly serve to interpret LES-results and to define the subgrid terms that have to be modeled. The usual subgrid terms are the commutator and the divergence of the turbulent stress, which is locally conservative by definition. As the sum of the two terms is conservative for a conservative filter, the adoption of a conservative subgrid model is a natural choice, which is in theoretical agreement with both the filtered and unfiltered Navier–Stokes equations.

Six subgrid models that involve explicit filter operations were investigated: Two dynamic eddy-viscosity models (ab), two filtering multiscale models (cd), a relaxation model (e) and a generalized similarity model (f). The dynamic models (ab) are obviously dissipative and momentum conserving for any test filter. However, a normalized and conservative (say self-adjoint) explicit filter is most consistent for the models (abf), since these models rely on the similarity assumption between basic and explicit filter level. Only then both filtered and modeled equations share the global conservation property.

In particular, the last four models (cdef) benefit from the incorporation of the adjoint filter. In this way they conserve momentum for each normalized nonuniform filter. In addition, due to the incorporation of the adjoint filter, models (cde) were analytically shown to dissipate kinetic energy, which is in agreement with the physical concept of the energy cascade process in three-dimensional turbulence.

Model (d), expressed by Eq. (27), is the most attractive one. Its divergence form corresponds to a symmetric nonlinear model of the turbulent stress tensor, formed from an arbitrary nonuniform explicit filter and the Smagorinsky model. The new model has the unusual, but desirable, combination of backscatter and an analytically positive dissipation. Model (f) explicitly incorporates a model for the commutator, but also the dissipative models may be interpreted to implicitly take into account the dissipative effects of both commutator and turbulent stress tensor.

Adjoint and self-adjoint nonuniform filters have been constructed, in continuous and discrete forms. The discrete adjoint can be calculated on a general unstructured grid. A compact, self-adjoint filter, applicable on orthogonal grids, has also been found.

In addition several smoothing properties of a filter were considered and from this point of view a positive, normalized and conservative filter function is preferable within the class of nonuniform kernel filters.

Non-kernel filters, like Laplace filters and projections were also discussed. A redefined, nonuniform Laplace filter is self-adjoint. Projection based methods in LES have recently gained increased attention.[5,20,24,28] In this paper, it was shown that an orthogonal projection operator is an example of a self-adjoint kernel filter. In this way, approaches based on projections can be integrated into the kernel-filtering approach.

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APPENDIX A: FILTER WIDTH DEFINITIONS

To define a filter width for a kernel filter, we first consider a filter in one dimension. The filter width is often defined by the second moment of the filter function:[3]

$$\frac{(\Delta(x))^2}{12}=\int_\Omega K_G(x,\xi)(\xi-x_m)^2 d\xi,$$
$$x_m=\int_\Omega K_G(x,\xi)\xi d\xi. \tag{A1}$$

If $G\geqslant 0$ and G is normalized, then the first equation is equal to the variance of the following local probability distribution function:

$$p_x(\xi)=K_G(x,\xi). \tag{A2}$$

Its mean equals x_m, which should be close to x.

Next, two simpler definitions are introduced, one involving the L_2-norm of the kernel,

$$\frac{1}{\Delta(x)}=\int_\Omega (K_G(x,\xi))^2 d\xi, \tag{A3}$$

and the other based on the central value of the filter function,

$$\frac{1}{\Delta(x)}=K_G(x,x). \tag{A4}$$

Both equations lead to the correct filter width for the standard top-hat filter. These definitions, possibly with a proportionality constant, can be useful if the second moment does not exist, which is, e.g., the case for the spectral cutoff.

Definition (A4) is the simplest expression. It results in

$$\Delta(x_i)=\frac{V_i}{\gamma}, \tag{A5}$$

for the three points filter (52). In the following, definition (A4) will be used to illustrate how the filter width of the conservative H and self-adjoint J in Sec. IV can be found. Take, for example, the nonuniform top-hat filter:

$$K_G(x,\xi)=\frac{1}{\Delta_G(x)} \text{ if } |x-\xi|<\frac{\Delta_G(x)}{2}, \tag{A6}$$

and zero elsewhere. Then the function $b(\xi)$ in Eq. (39) can be expressed as

$$b(\xi)=\int_{y_1}^{y_2}\frac{1}{\Delta_G(y)}dy, \tag{A7}$$

where

$$y_1+\frac{\Delta_G(y_1)}{2}=\xi=y_2+\frac{\Delta_G(y_2)}{2}, \tag{A8}$$

has to be solved for each ξ. The filter widths of H and J equal

$$\Delta_J(x)=\Delta_H(x)=\frac{1}{K_H(x,x)}=\frac{\Delta_G(x)}{1+(1-b(x))\Delta_G(x)/V}, \tag{A9}$$

which is easily recognized as a correction to $\Delta_G(x)$, the filter width of G.

Finally, we define the filter width for the two discrete (nonorthogonal) three-dimensional filters introduced in Sec. IV. Suppose Eqs. (45) or (46) represents an explicit filter, adopted for one of the subgrid models in Sec. III. In volume i, the corresponding local filter width, say $\hat{\Delta}_i$, is equal to

$$\hat{\Delta}_i=\left[\sum_{j\in B_i} V_j\right]^{1/3}. \tag{A10}$$

In cases the basic filter width (of the implicit grid-filter) is assumed to be equal to the local grid size,

$$\Delta_i=V_i^{1/3}, \tag{A11}$$

can be used in volume V_i, assuming the control volumes form a partioning of Ω.

APPENDIX B: SMOOTHING PROPERTIES

In the case of nonuniform kernel filters, a positive[16] and normalized kernel K_G is required to prove the smoothing properties (60) and (61):

$$Gf(x)=\int_\Omega G(x,\xi)f(\xi)d\xi\leqslant\int_\Omega G(x,\xi)\ \ \max_\Omega(f)d\xi$$
$$=\max_\Omega(f). \tag{B1}$$

Application of this equation to $-Gf$ yields

$$Gf\geqslant-\max_\Omega(-f)=\min_\Omega(f), \tag{B2}$$

and inequality (60) is thus satisfied. Positivity of a normalized nonuniform kernel filters implies a positive trace of the turbulent stress tensor and similarly a positive variance $\mathrm{var}_f(x)$ of a function $f(x)$ (see Ref. 15). If the filter is conservative as well (e.g., self-adjoint) then

$$\int_\Omega(f(x))^2dx=\int_\Omega[(Gf(x))^2+\mathrm{var}_f(x)]dx$$
$$\geqslant\int_\Omega(Gf(x))^2dx. \tag{B3}$$

Evidently, inequality (61) holds.

Finally, we show that the second smoothing property (61) is also valid for an orthogonal projection operator. If the orthogonal basis functions are normalized then

$$\int_\Omega Gf^2dx=\Sigma_{k=1}^N\alpha_k^2\leqslant\int_\Omega f^2dx, \tag{B4}$$

which relies on Parseval's identity.

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